数理解析研究所講究録
第 1702 巻 2010 年 149-157

Existence of multiple spike stationary patterns in a chemotaxis model with weak saturation

首都大学東京大学院理工学研究科　倉田和浩 (Kazuhiro Kurata)
Department of Mathematics and Information Sciences,
Tokyo Metropolitan University

1 Introduction

In this paper, we present recent results obtained by a joint work with Dr. Kotaro Morimoto (see [4] for the details) with a brief outline of the strategy of the proof.

We consider a stationary problem to the following chemotaxis model:

$$P_t = D\nabla\cdot(\nabla P - P\nabla\phi(W)),\ (x,t)\in\Omega\times(0,T)$$

$$W_t = \epsilon^2\Delta W + F(P,W),\ (x,t)\in\Omega\times(0,T),$$
$$\frac{\partial P}{\partial\nu} = \frac{\partial W}{\partial\nu} = 0,\ (x,t)\in\partial\Omega\times(0,T),$$

where $\Omega\subset\mathbf{R}^N$ is a bounded smooth domain, and $D,\epsilon^2>0$ are positive constants. Here, $P(x,t)$ is a population density and $W(x,t)$ is a density of certain chemicals, $\phi(W)$ is a sensitivity function, and $F(P,W)$ is a certain kinetic reaction term.

Formally, from the first equation it follows from the boundary condition that

$$\begin{aligned}\frac{d}{dt}\Big(\int_\Omega P(x,t)\,dx\Big) &= \int_\Omega\frac{\partial P}{\partial t}\,dx = D\int_\Omega\nabla\cdot(\nabla P - P\nabla\phi(W))\,dx\\ &= D\int_{\partial\Omega}\nu\cdot(\nabla P - P\nabla\phi(W))\,dS = 0\end{aligned}$$

holds. Thus we have

$$\int_\Omega P(x,t)\,dx = \lambda\ (\text{constant}). \tag{1}$$

As the stationary problem to this model, we have

$$0 = D\nabla\cdot(\nabla P - P\nabla\phi(W)) = D\nabla\cdot(P\nabla(\log P - \phi(W))),\ x\in\Omega,$$

$$0 = \epsilon^2\Delta W + F(P,W),\ x\in\Omega,$$
$$\frac{\partial P}{\partial\nu} = \frac{\partial W}{\partial\nu} = 0,\ x\in\partial\Omega.$$

Then multiplying $\log P - \phi(W)$ and integration by parts, we have

$$\int_\Omega P|\nabla(\log P - \phi(W))|^2\,dx = \int_{\partial\Omega}(\frac{\partial P}{\partial\nu} - P\phi'(W)\frac{\partial W}{\partial\nu})\,dS = 0.$$

Hence, $\log P - \phi(W) = C$ (constant). From the L^1 conservation law (1), we may assume $\int_\Omega P(x)\,dx = 1$. So, We obtain

$$P(x) = \frac{e^{\phi(W)}}{\int_\Omega e^{\phi(W)}\,dx}.$$

Now we choose $\phi(W) = p\log W$ for some $p > 0$, although other functions $\phi(W)$ satisfying $\phi'(W) > 0$ can be chosen, e.g. $\phi(W) = aW, \phi(W) = \frac{aW}{1+W}$, etc. Then , we have

$$P(x) = \frac{W^p}{\int_\Omega W^p\,dx}.$$

Therefore, we obtain the following problem for W:

$$0 = \epsilon^2\Delta W + F(\frac{W^p}{\int_\Omega W^p\,dx}, W),\ x\in\Omega,\quad \frac{\partial W}{\partial\nu} = 0,\ x\in\partial\Omega. \tag{2}$$

As a kinetic reaction term $F(P,W)$, there are several models choosing the linear growth: $F(P,W) = P - \mu W$ with $\mu > 0$ (see e.g. [5]), or choosing the exponential growth model:$F(P,W) = (P-\mu)W$: (see Levin-Sleeman([6])), or the saturation growth model:

$$F(P,W) = \frac{PW}{1+\nu W)} - \mu W + \gamma\frac{P}{1+P}$$

with $\nu,\gamma > 0$ (see Othmer-Stevens([8])).

In this paper, we consider two cases which have a sturation effect:

$$F_1(P,W) = -W + \frac{PW^q}{\alpha+\gamma W^q}, \alpha,\gamma,q > 0;$$

$$F_2(P,W) = -W + \frac{P}{1+kP}, k > 0.$$

For $F(P,W) = F_1(P,W)$ (or $F(P,W) = F_2(P,W)$), the problem (2) become as follows:

$$0 = \epsilon^2\Delta W - W + \frac{1}{\int_\Omega W^p\,dx}\Big(\frac{W^{p+q}}{\alpha+\gamma W^q}\Big),\ x\in\Omega, \tag{3}$$

$$0 = \epsilon^2\Delta W - W + \frac{1}{\int_\Omega W^p\,dx}\Big(\frac{W^p}{1+k\frac{W^p}{\int_\Omega W^p\,dx}}\Big),\ x\in\Omega, \tag{4}$$

under the Neumann boundary condition $\frac{\partial W}{\partial\nu} = 0,\ x\in\partial\Omega$. We are concerned with the point condensation phenomena for these system and studied the existence of multiple spike stationary solutions to (3) or (4) on an axially symmetric smooth domain Ω under certain weak saturation conditions on the parameters α,γ and k (see the next section for the precise assumptions).

For other interesting phenomena (including finite-time blow-up phenomena) on these system, see [8], [6],[9] and the references therein.

2 Main Results

We assume that Ω is x_N-axially symmetric when $N \geq 2$ and the parameters p, q, k, α and γ satsify the following assumptions:

Assumptions:

$(A.0)$: When $N \geq 2$, Ω is symmetric w.r.t. x_N-axis, i.e. if $x = (x', x_N) \in \Omega$, then $(x'', x_N) \in \Omega$ for any $x = (x'', x_N)$ with $|x'| = |x''|$. Moreover, $1 < p < +\infty$ for $N = 1, 2$ and $1 < p < \frac{N+2}{N-2}$ for $N \geq 3$.

$(A.1)$: $q > 0, \alpha = \alpha(\epsilon), \gamma = \gamma(\epsilon) > 0$ and there exists $\alpha_0 \in [0, +\infty)$ s.t.

$$\lim_{\epsilon \to 0} \epsilon^N (\alpha \gamma^{q-1})^{\frac{1}{q}} = \alpha_0.$$

$(B.1)$: $k = k(\epsilon) > 0$ and there exists $k_0 \in [0, +\infty)$ s.t.

$$\lim_{\epsilon \to 0} \epsilon^{-N} k = k_0.$$

We say the model has a **weak saturation effect** if the parameters α, γ or k satisfy the condition $(A.1)$ or $(B.1)$ by the analogy to the condition given to the Gierer-Meinhardt system (see [10] and [3]). Although up to now this definition rather comes from the technical point of view to obtain spiky stationary solutions, when these conditions does not hold (we call the model with **strong saturation**), other type stationary solution may occur, for example solutions with stripe patterns or solutions with inner transition layers.

2.1 main result for $F_1(P, W) = -W + \frac{PW^q}{\alpha + \gamma W^q}$

Let $F(P, W) = F_1(P, W)$ and assume $(A.0), (A.1)$. Let $P_1, P_2, \cdots, P_{2n}$ be the intersection points of $\partial\Omega$ and the x_N-axis. We choose m points $Q_k = P_{j_k} (k = 1, 2, \cdots, m)$ from $\{P_j\}_{j=1}^{2n}$ arbitrarily.

Theorem 1 *Then there exists $\alpha_1 \in (0, +\infty]$ s.t. if $0 \leq \alpha_0 < \alpha_1$, then for sufficiently small $\epsilon > 0$ there exists a stationary solution (P_ϵ, W_ϵ) to the chemotaxis model satisfying $P_\epsilon = \frac{u_\epsilon^p}{\int_\Omega u_\epsilon^p \, dx}$, $W_\epsilon = \frac{u_\epsilon}{\gamma \int_\Omega u_\epsilon^p \, dx}$. Here, u_ϵ is a solution to*

$$\epsilon^2 \Delta u - u + \frac{u^{p+q}}{\alpha \gamma^{q-1} (\int_\Omega u^p \, dx)^q + u^q} = 0, \; x \in \Omega,$$

$$\frac{\partial u}{\partial \nu} = 0 \;\; (x \in \partial\Omega), \quad u(x) > 0 \;\; (x \in \Omega)$$

satisfying the following asymptotic profile:

$$(1) \quad u_\epsilon(x) \sim w_{\delta_*}(\frac{x - Q_j}{\epsilon}), \; x \sim Q_j;$$

$$(2) \quad \int_{\Omega} u_\epsilon^p\, dx = \frac{m\epsilon^N}{2} \int_{\mathbf{R}^N} w_{\delta_*}^p\, dx + o(\epsilon^N);$$

$$(3) \quad u_\epsilon(x) \le C \exp(-\frac{Cd(x,Q)}{\epsilon}),\ x \in \Omega,\ Q = \{Q_1, \cdots, Q_m\}.$$

Remark 1 *w_δ ($\delta \ge 0$) is a unique positive solution to*

$$\Delta w - w + \frac{w^{p+q}}{\delta + w^q} = 0,\ x \in \mathbf{R}^N,$$

$$w(0) = \max w, \quad w(x) \to 0\ (|x| \to +\infty).$$

Moreover, δ_* is determined for α_0 as follows:

$$\frac{\delta_*^{1/q}}{\frac{m}{2}\int_{\mathbf{R}^N} w_{\delta_*}^p\, dx} = \alpha_0.$$

Remark 2 *Furthermore, under the assumption $p + q < +\infty$ ($N = 1,2$), $p + q < \frac{N+2}{N-2}$ ($N \ge 3$), we have an information for $\alpha_1 \in (0, +\infty]$ as follows: for $q = 1$, $\alpha_1 \ge (\frac{m}{2}\int_{\mathbf{R}^N} v_0^p\, dx)^{-1}$; for $q > 1$, $\alpha_1 = +\infty$; for $0 < q < 1$, $0 < \alpha_1 < +\infty$. Here, v_0 is a unique positive solution to*

$$\Delta v - v + v^{p+q} = 0,\ x \in \mathbf{R}^N, \quad v(0) = \max v,\ v(x) \to 0\ (|x| \to +\infty).$$

2.2 main result for $F_2(P, W) = -W + \frac{P}{1+kP}$

Let $F(P, W) = F_2(P, W)$ and assume $(A.0), (B.1)$. Choose arbitrary m points $Q_j (j = 1, 2, \cdots, m)$ as before.

Theorem 2 *Then for sufficiently small $\epsilon > 0$ there exists a stationary solution (P_ϵ, W_ϵ) to the chemotaxis model satisfying $P_\epsilon = \frac{u_\epsilon^p}{\int_\Omega u_\epsilon^p\, dx}$, $W_\epsilon = \frac{u_\epsilon}{\int_\Omega u_\epsilon^p\, dx}$. Here, u_ϵ is a solution to*

$$\epsilon^2 \Delta u - u + \frac{u^p}{1 + k(\int_\Omega u^p\, dx)^{-1} u^p} = 0,\ x \in \Omega,$$

$$\frac{\partial u}{\partial \nu} = 0\ \ (x \in \partial\Omega), \quad u(x) > 0\ \ (x \in \Omega)$$

satisfying the following asymptotic profile:

$$(1) \quad u_\epsilon(x) \sim w_{\delta_{**}}(\frac{x - Q_j}{\epsilon}),\ x \sim Q_j;$$

$$(2) \quad \int_{\Omega} u_\epsilon^p\, dx = \frac{m\epsilon^N}{2} \int_{\mathbf{R}^N} w_{\delta_{**}}^p\, dx + o(\epsilon^N);$$

$$(3) \quad u_\epsilon(x) \le C \exp(-\frac{Cd(x,Q)}{\epsilon}),\ x \in \Omega,\ Q = \{Q_1, \cdots, Q_m\}.$$

Remark 3 *There exists a constant δ^* such that for $\delta \in [0, \delta^*)$ w_δ is a unique positive solution to*

$$\Delta w - w + \frac{w^p}{1 + \delta w^p} = 0, \ x \in \mathbf{R}^N,$$
$$w(0) = \max w, \quad w(x) \to 0 \ (|x| \to +\infty).$$

Moreover, δ_{**} is determined for k_0 as follows:

$$\delta_{**} \int_{\mathbf{R}^N} w_{\delta_{**}}^p \, dx = k_0.$$

3 Related works

When $k = 0$, or $\alpha = 0$, or $\gamma = 0$, then the problem is decoupled and is reduced to the study of the equation $\Delta u - u + u^p = 0$(e.g. Lin-Ni-Takagi([5]), Ni-Takagi([7])).
Sleeman,Ward, Wei([9]) studied the case $F(P, W) = F_1(P, W)$ with $q = 1$ and the fixed $\alpha = 1$, i.e. $\alpha_0 = 0$. They showed existence of one spike solution which concentrates at the nondegenerate local maximum point of the mean curvature function on $\partial\Omega$ and studied its stability. Although our results are restricted for the domains with axial symmetry, our results do note need nondegeneracy of the mean curvature function, e.g. a ball or an annulus are allowed as Ω.
Similar problem for the Gierer-Meinhardt (shadow) system have been studied by Wei-Winter([10]), Kurata-Morimoto([3]).

4 Uniqueness and nondegeneracy of solutions to the limiting problem

To construct multi-spike stationary solutions in main theorems, we need good approximated solutions. To construct such approximated solutions, the analysis of the corresponding limiting problems is very important.
There are many studies on the uniqueness and nondegeneracy of solutions to the following problem:

$$\Delta w + g(w) = 0, \ w > 0 \ in \ \mathbf{R}^N, \quad w(0) = \max_{\mathbf{R}^N} w, \quad w(z) \to 0 \ as \ |z| \to \infty. \tag{5}$$

We present useful well-known conditions to assure the uniqueness and the nondegeneracy. We always assume $g \in C^1([0, \infty))$, and we define

$$G(v) := \frac{vg'(v)}{g(v)}.$$

Definition 1 *We say g is Type A if g satisfies the following conditions:*
(g1) $g(0) = 0$, $g'(0) < 0$, and there exists $a > 0$ such that $g(a) = 0$, $g(v) < 0$ for

$v \in (0, a)$, and $g'(a) > 0$.
(g2) There exists $\theta > a$ such that $\int_0^\theta g(t)dt = 0$ and $g(v) > 0$ for $v \in (a, \theta)$.
(g3A) $g(v) > 0$ for $v > a$.
(g4A) The function $G(v)$ is nonincreasing in $[\theta, \infty)$ and converges to a finite limit $L \geq 1$ as $v \to \infty$. $G(v) \geq G(\theta)$ for $v \in (a, \theta]$. $G(v) \leq L$ for $v \in (0, a)$.
(g5A) It holds that $\lim_{v \to \infty} g(v)/v^l = 0$ for some $l \in [0, \infty)$ in case $N = 1, 2$, $l \in [0, \frac{N+2}{N-2})$ in case $N \geq 3$.

Definition 2 *We say g is Type B if $g(v)$ satisfies the following conditions in addition to (g1) and (g2):*
(g3B) There exists $b > \theta$ such that $g(b) = 0$, $g(v) > 0$ for $v \in (a, b)$, $g(v) < 0$ for $v > b$.
(g4B) Let $\rho \in [a, b)$ be the smallest number such that $(v - \rho)g'(v) \leq g(v)$ for $v \in (\rho, b)$, then either (i) or (ii) holds:
(i) $\theta \geq \rho$,
(ii) $\theta < \rho$ and that $G(v)$ is nonincreasing in (θ, ρ), $G(v) \geq G(\theta)$ for $v \in (a, \theta)$, $G(v) \leq G(\rho)$ for $v \in (0, a) \cup (\rho, b)$.

Proposition 1 *If g is Type A or B, then (5) has a unique solution w, and it satisfies the following:*
(a) $w \in C^2(\mathbf{R}^N) \cap H^2(\mathbf{R}^N)$.
(b) w is radially symmetric, i.e., $w(x) = w(|x|)$, and $w'(r) < 0$ for $r = |x| > 0$.
(c) w decays exponentially together with its derivatives up to the order of 2, that is, for any $|\alpha| \leq 2$, $|D^\alpha w(x)| \leq Ce^{-c|x|}$, $x \in \mathbf{R}^N$, holds for some $C, c > 0$.
(d) The linearized operator $L = \Delta + g'(w)$ on $L^2(\mathbf{R}^N)$ with domain $H^2(\mathbf{R}^N)$ satisfies $Ker(L) = span\{\frac{\partial w}{\partial x_1}, \cdots, \frac{\partial w}{\partial x_N}\}$, and if we regard L as an operator on $L^2_{rad}(\mathbf{R}^N)$ with $Dom(L) = H^2_{rad}(\mathbf{R}^N)$, then L is bijective, where $L^2_{rad}(\mathbf{R}^N)$ and $H^2_{rad}(\mathbf{R}^N)$ stand for the restricted spaces of $L^2(\mathbf{R}^N)$ and $H^2(\mathbf{R}^N)$ to the radially symmetric function spaces, respectively.

For the details of Proposition 1, see [2],[1] and the references therein. We can show the nonlinearities $g(t)$ treated in this paper are type A or Type B. Therefore, by using Proposition 1, we obtain the following results which are very important in our analysis.

Proposition 2 *Assume that, $q > 0$, $1 < p < (N+2)/(N-2)$ if $\delta = 0$ and $N \geq 3$, $1 < p < \infty$ if $\delta > 0$ or $N = 1, 2$. Then for any $0 < \delta < +\infty$ and $g(t) = -t + \frac{t^{p+q}}{\delta + t^q}$, the existence, uniqueness and nondegeneracy of ground state holds.*

Proposition 3 *Assume that, $1 < p < (N+2)/(N-2)$ if $\delta = 0$ and $N \geq 3$, $1 < p < \infty$ if $\delta > 0$ or $N = 1, 2$. Then there exists $\delta^* > 0$ such that, if $0 < \delta < \delta^*$ and $g(t) = -t + \frac{t^p}{1+\delta t^p}$, the existence, uniqueness and nondegeneracy of the ground sate holds.*

For the proof of Proposition 2, 3, see [4]. Proposition 2 and 3 for the case $\delta = 0$ is well-known and Proposition 3 for the case $\delta > 0$ and $p = 2$ is due to [10]. However, we emphasize that Proposition 2 and 3 for other cases seems new.

5 Strategy and Outline of the Proof

We just briefly explain our strategy and te outline of the proof of our theorems.

• **(STEP1)** For $F = F_1$, let us define $u(x)$ by $W(x) = \frac{u}{\gamma \int_\Omega u^p\,dx}$. Then the problem is reduced to find the pair $(u(x), \delta)$ s.t.

$$\epsilon^2 \Delta u - u + \frac{u^{p+q}}{\delta + u^q} = 0, \ x \in \Omega$$

with Neumann BC and

$$\delta = \alpha \gamma^{q-1} \left(\int_\Omega u^p\,dx\right)^q.$$

• For $F = F_2$, let us define $u(x)$ by $W(x) = \frac{u}{\int_\Omega u^p\,dx}$. Then the problem is reduced to find the pair $(u(x), \delta)$ s.t.

$$\epsilon^2 \Delta u - u + \frac{u^p}{1 + \delta u^p} = 0, \ x \in \Omega$$

with Neumann BC and

$$\delta = k \left(\int_\Omega u^p\,dx\right)^{-1}.$$

• **(STEP2)** For fixed $\delta > 0$, construct a x_N-axially symmetric solution $u = u_{\delta,\epsilon}(x) = U_{\delta,\epsilon}(x) + \epsilon \phi_{\delta,\epsilon}(x)$ which is continuous in the parameter $\delta \geq 0$, where $U_{\delta,\epsilon}$ is a suitable approximated solution and a uniformly bounded function $\phi_{\delta,\epsilon}(x)$.
In particular, *key points for (STEP2) in our analysis are as follows*:
(a) For the special nonlinearities above, we can obtain *the uniqueness and nondegeneracy of the ground state.*
(b) Then, the assumption on the symmetry of domain implies the invertibility of the linearized operator $L_\delta = \Delta - 1 + f_\delta'(U_{\delta,\epsilon})$ at the approximated solution $U_{\delta,\epsilon}$ on the Sobolev spaces with axially symmetry, e.g. for the case $F = F_1$ we have

$$f_\delta(u) = \frac{u^{p+q}}{\delta + u^q}.$$

(c) Careful uniform estimates with respect to the parameter δ and the contraction mapping principle with parameter enable us to obtain a suitable solution which is continuous with respect to the parameter δ.

•**(STEP3)** Using a global estimate for $u_{\delta,\epsilon}(x)$ (spiky profile) and an asymptotic behaviour of integrals for $\int_\Omega u^p\,dx$, find the $\delta = \delta_\epsilon$ satisfying the matching condition (for $F = F_1$):

$$(\delta_\epsilon)^{1/q} \left(\int_\Omega u_{\delta,\epsilon}^p\,dx\right)^{-1} = (\alpha \gamma^{q-1})^{1/q}$$

Now, it is important to know the behaviour of the function

$$\beta(\delta) = \frac{\delta^{1/q}}{\int_{\mathbf{R}^N} w_\delta^p\,dx}.$$

The matching condition, e.g. for the case $F = F_2$, is as follows:

$$\delta_\epsilon \left(\int_\Omega u_{\delta,\epsilon}^p \, dx \right) = k.$$

Here, the behaviour of the function $\beta(\delta) = \delta \int_{\mathbf{R}^N} w_\delta^p \, dx$ is important.

6 Summary and Problems

We obtained mutiple spike stationary solutions to ceratian chemotaxis model with weak saturation on x_N-axially symmetric domains. In particular, we showed uniqueness and nondegenracy of ground state for associated scalar equations and gave a systematic method to constract a solution of a ceratin class of nonlinear ellptic equations with nonlocal terms.

We note the following as future Problems:

1. How about the following case?

$$F(P, W) = \frac{PW}{k + W} - \mu W + \gamma \frac{P}{1 + P}$$

2. Construct multiple spiky solutions for general domains and study stability of spiky solutions. See [9] the stablity of one peak solution which concentrates near the most curved part of the boundary $\partial\Omega$ for the case $F = F_2$ with $\alpha = 1$. Construction of multiple spiky solutions for general domains without any symmetry seems more difficult.

3. The notion of the **weak saturation effect** to these chemotaxis models was proposed for the first time in [4]. How about the case with strong saturation effect?

Acknowledgement.
The author is thankful to Professor Tetsutaro Shibata for inviting me to this conference.

参考文献

[1] P. Bates, E.N. Dancer and J. Shi, Multi-spike stationary solutions of the Cahn-Hilliard equation in higher-dimension and instability, Adv. Differential Equations, 4(1999), 1-69.

[2] P. Bates and J. Shi, Existence and instability of spike layer solutions to singular perturbation problems, J. Functional Analysis, 196(2002), 211-264.

[3] K. Kurata and K. Morimoto, Construction and asymptotic behavior of multi-peak solutions to the Gierer-Meinhardt system with saturation, Commun. Pure Appl. Anal., 7 (2008) 1443-1482.

[4] K. Kurata, K. Morimoto, Existence of multiple spike stationary patterns in a chemotaxis model with weak saturation, submitted.

[5] C.S. Lin ,W.M. Ni and I. Takagi, Large amplitude stationary solutions to a chemotaxis system, J. Diff. Equations, 72(1988), 1-27.

[6] H.A.Levine and B.D.Sleeman, A system of reaction diffusion equation arising in the theory of reinforced random walks, SIAM J. Appl. Math., 57 (1997) 683-730.

[7] W.-M. Ni and I. Takagi, Point condensation generated by a reaction-diffusion system in axially symmetric domains, Japan J. Indust. Appl. Math., 12 (1995) 327-365.

[8] H.G.Othmer and A.Stevens, Aggregation, blowup, and collapse: the ABC's of taxis in reinforced random walks, SIAM J. Appl. Math., 57 (1997) 1044-1081.

[9] B. D. Sleeman, M. J. Ward and J. Wei, The existence and stability of spike patterns in a chemotaxis model, SIAM J. Appl Math., 65 (2005) 790-817.

[10] J. Wei and M. Winter, On the Gierer-Meinhardt system with saturation, Commun. Contemp. Math., 6(2004), 259-277.